JOURNAL OF FUNCTIONAL ANALYSIS 75, 58–91 (1987)

Prequantization of Infinite Dimensional Dynamical Systems

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Communicated by the Editors

Received June 12, 1985

The problem of prequantization of infinite dimensional dynamical systems is considered, using a Gaussian measure on an Abstract Wiener Space to play the role of volume element replacing the Liouville measure. As an example, it is shown that the flow on $L_1^2(\Omega) \times L_0^2(\Omega)$ corresponding to the nonlinear Klein–Gordon field over a two dimensional static space time $N = \Omega \times \mathbb{R}$ leaves the Gaussian measure defined on this space w.r.t. the norm on $L_{s+1}^2 \times L_s^2$ (where $\frac{1}{2} < s < 1$) quasi-invariant. This makes it possible to carry out the prequantization in this case.

1. INTRODUCTION

In two papers in the early sixties [25, 27], I. E. Segal argued heuristically that a procedure for quantizing classical mechanical systems on (finite dimensional) manifolds would, when applied to the "solution manifold" of a nonlinear relativistic field equation such as the Klein–Gordon equation,

$$\Box\, u - m^2 u = -F'(u), \tag{1.1}$$

yield a local, relativistically covariant quantum field theory satisfying the Wightman axioms. The quantization procedure sketched in [25] can be paraphrased as follows:

> Let (M, ω) be a symplectic manifold of dimension $2n$ and let $\mathbf{H} = L_2(M, \omega^n) \otimes \mathbb{C}$, the Hilbert space of complex-valued functions on M, square integrable w.r.t. ω^n, the Liouville measure on M defined by ω. Then to $f \in C^\infty(M, \mathbb{R})$ assign an operator $\rho(f)$ on $\mathbf{H}$ by
>
> $$\rho(f)\varphi = (-ihX_f\varphi - \vartheta \rfloor X_f)\varphi \tag{1.2}$$
>
> [25, p. 476], where ϑ satisfies $\omega = d\vartheta$, i.e., ϑ is a symplectic potential for ω.

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0022-1236/87 $3.00

The above procedure is, of course, nothing other than what is now known as prequantization (minus, however, the term $f\varphi$; cf. (1.2′)), which is a part of the Geometric Quantization (GQ) program later developed by (among others) Souriau, Blattner, Kostant and Sternberg. We will here use as reference on GQ the book by Woodhouse [32].

The basic data which determine a quantum structure on a symplectic manifold M are, loosely speaking, the symplectic form ω and some extra condition which determines a complex structure in each tangent space such as a Riemannian metric or a Lagrangian subbundle of $TM \otimes \mathbb{C}$ (a polarization in the terminology of GQ).

In a number of papers, Segal and others have studied the quantization data determined by simple relativistic field equations and it turns out that such structures are indeed defined in a natural way by these equations. See, for example, [28, 19, 20]. The main problem left open by this work is to find a replacement for the Liouville volume form ω^n in the infinite dimensional case. This is the problem which is central to this paper.

The crucial property of ω^n is that it is left invariant by flows of Hamiltonian vector fields,

$$\Phi^*_{X,t}\omega^n = \omega^n,$$

which implies that $\Phi_{X,t}$ lifts to a group of unitary transformations on $L^2(M, \omega^n) \otimes \mathbb{C}$. This property was first utilized by Koopmans to get unitary representations of symplectic transformations.

In the infinite dimensional case there is no such thing as an invariant volume form and one has to be content to work with a quasi-invariant measure, i.e., one which is transformed into an equivalent measure by the relevant transformations. This leads to a modification of the "prequantization rule" (1.2) into

$$\rho(f)\varphi = -ihX_f\varphi - (\vartheta \rfloor X_f)\varphi - ih\,\tfrac{1}{2}\,\delta(X_f)\varphi + f\varphi, \tag{1.2′}$$

where δ is an analogue of the ordinary divergence, i.e., the logarithmic derivative of a volume form, defined w.r.t. the particular measure used. See [32, Sect. 5.10] for a discussion of formulas of this type. It is clear that if δ satisfies the usual relation

$$\delta([X, Y]) = X\,\delta(Y) - Y\,\delta(X),$$

then the map $f \to \rho(f)$ satisfies the canonical commutation relations (see Section 2). Thus the definition of the prequantization (1.2′) depends essentially on the existence and regularity of $\delta(X)$ for Hamiltonian vector fields X corresponding to the relevant observables.

The formula (1.2′) can be compared with the "operator representation of the Lie algebra of Poisson brackets" studied in a series of papers by Raczka and others [2, 3, 22] and which in a slightly different notation is given by a map

$$f \to \hat{f} = -iX_f - \langle df(x), x \rangle + f, \tag{1.3}$$

(cf. [3, (2.7)]). It is possible to show that (1.3) defines a representation of the canonical commutation relations as operators on a certain Frechet space, but there is no natural Hilbert space in this framework and so the $\hat{f}$ will not be selfadjoint operators. Thus, as mentioned in [3, p. 301], the missing ingredient is an invariant measure. However, it appears clear that the best one can hope for is a quasi-invariant measure, which makes some kind of divergence term necessary.

The method which seems most natural for implementing this scheme is the theory of nonlinear transformations of Gaussian measures and integration on infinite dimensional manifolds developed by Gross [11], Kuo [15, 16], Ramer [23], Piech [21] and others. This method turns out (as one should expect from other approaches to the same problem) to have fundamental limitations which do not allow us to deal directly with, for example, a $\lambda\varphi_4^4$-theory but only with a regularized $\lambda(R\varphi)_4^4$-theory, where R is some smoothing operator.

This is similar to the situation in most, if not all schemes for field-quantization, that for space dimension $n > 1$, the representation of the interacting system is reached only after a limiting procedure from a regularized version.

In one space dimension it turns out, however, that in an appropriate analytical setting it is possible to prove that the transformation induced by a flow corresponding to the Klein–Gordon equation preserves the measure class of a certain Gaussian measure. This possibility was indicated by Segal in [29], based on the observation that the scattering transformation S in the case of one space dimension satisfies $S = I + K$ with $DK \in HS(H)$, where H is a certain Sobolev space. The work in [29] was in the context of generalized Gaussian measures where there does not exist a method to handle nonlinearities of the present type. It was also observed in [29] that for more than one space dimension, DK will *not* be of Hilbert–Schmidt type, which means that some regularization is necessary for the present method to be applicable.

The problem of convergence for $n > 1$ is left open here. Further, since we are dealing only with prequantization (i.e.; no polarization) we do not, as is well known, get a "correct" representation (in particular one cannot expect a prequantization representation of the basic position and momentum observables to be irreducible). Finally, the question of relativistic covariance of the resulting representation of the Klein–Gordon flow is not studied.

To motivate this work we should therefore remark that, apart from the (possible) intrinsic mathematical interest in studying the geometry of the prequantization representation in the infinite dimensional case, there is a fundamental conceptual difference between the "geometric" and the "canonical" quantization procedures.

The difference is in the way the classical dynamics is presented in the quantum version of the system and can be described as follows: In the "geometric" procedure one gets the quantum dynamics by "lifting" the classical dynamics to a certain bundle over the phase space of the classical system, the bundle of half-forms, while in the "canonical procedure" one constructs the quantum Hamiltonian using an "ordering rule." Thus, while one expects both procedures to give the same answer on all problems where they are defined, GQ may be used as a tool to study the properties of the quantization problem in a different way than canonical quantization.

The facts that GQ in the finite dimensional case is an elegant procedure which is well defined in such cases as constrained systems, where it is difficult to give meaning to, for example, the canonical procedure, that it is related in a deep way to the Feynman-integral and that it gives good insight into the global properties of quantization seems to motivate an attempt to extend it to field theories. This paper represents a first step in that direction.

The contents of this paper can be outlined as follows. In Section 2 the quantization problem is discussed in general terms, stressing the points of similarity between the case of finite dimensional theories and Boson fields. As a preliminary to the work in the rest of the paper, an outline of the prequantization procedure in the case with a noninvariant volume form is given.

In Sections 3 and 4 some preliminaries on Abstract Wiener Spaces and nonlinear transformations of Gaussian measures are given for the convenience of the reader. This is then applied in Sections 5–8 to derive the technical results which are necessary for carrying out the program outlined in Section 2. Finally, in Section 9, these tools are used to give a prequantization of the Klein–Gordon equation. The main result can be summarized as follows:

THEOREM. *In a spatially compact static two dimensional spacetime, the flow corresponding to the nonlinear Klein–Gordon equation preserves the measure class of a certain Gaussian measure, and lifts to a strongly continuous unitary group on a corresponding Hilbert space.*

Analogous results hold for a regularized version of the Klein–Gordon equation in higher dimensional spacetimes.

2. Background on Quantization

As a preliminary for the rest of the paper, we collect here some remarks on the quantization problem and some similarities between the finite and infinite dimensional situations.

It should be stated clearly that many important concepts and problems relevant to the field quantization problem are ignored, partly to simplify the discussion, partly due to the ignorance of the author.

Let (X, ω) be a symplectic manifold. The problem of quantization may in general be thought of as the problem of finding a representation, i.e., a map $f \to \rho(f)$ of a set of classical observables (functions in some space which we will denote by $C^\infty(X, \mathbb{R})$) on a X as Hermitian operators on a representation Hilbert space $\mathbf{H}$. The representation should satisfy the following basic axioms known as the "canonical commutation relations":

$$[\rho(f), \rho(g)] = \rho(\{f, g\}) \tag{Q1}$$

$$\rho(1) = I_{\mathbf{H}} \tag{Q2}$$

$$\textit{irreducibility}. \tag{Q3}$$

In the finite dimensional case $\mathbf{F} = C^\infty(X, \mathbb{R})$ with product given by the Poisson bracket $\{\ ,\ \}$ forms a Lie-algebra. The Poisson bracket satisfies the Jacobi-identities and also the derivation rule which in terms of the representation ρ corresponds to

$$[\rho(fg), \rho(h)] = \rho(f)[\rho(g), \rho(h)] + \rho(g)[\rho(f), \rho(h)]. \tag{Q4}$$

These algebra properties of $\mathbf{F}$ were taken as fundamental for the quantization problem by Dirac and the problem of finding an isomorphism from $\mathbf{F}$ into $\mathbf{A}(\mathbf{H})$ (the algebra of self-adjoint operators on $\mathbf{H}$) satisfying (Q1)–(Q4) become known as the Dirac problem. It has been shown that in general this problem has no solution and therefore at least one of the axioms (Q1)–(Q4) has to be modified. See [5] for a discussion of this.

One general approach to the problem of specifying a quantization rule is given by Geometric Quantization, which is a developement of ideas by van Hove, Segal and Bargmann among others. It is the "first step" in this procedure, known as prequantization, that will be studied here.

Before going into the definitions which will be used, it might be interesting to make a few remarks on the formal similarities that exist between the problem of quantizing a finite dimensional classical system and a (Bosonic) field theory. In both cases canonical commutation relations have to be satisfied. In the case of field theory there is the additional problem of preserving relativistic invariance, which, however, can be satisfied in the linear case by requiring that the quantum data are invariantly defined.

Let (X, ω) be a real $2n$ dimensional symplectic vector space. Then X is symplectically isomorphic to $T^*\mathbf{R}^n$ with the canonical symplectic form. The quantization problem can in the linear case be described as follows (see [14]).

Let ξ be some real one dimensional vector space and let z be a fixed element of ξ. Define the Heisenberg Lie-algebra $\mathbf{n} = T^*\mathbb{R}^n \times \xi$ by taking ξ as the center of $\mathbf{n}$ and defining a product

$$[f, g] = \{f, g\} z. \tag{2.1}$$

Let N, the Heisenberg group, be the simply connected Lie group having $\mathbf{n}$ as its Lie algebra. The Stone–von Neumann theorem states that any irreducible representation of N as unitary operators on a Hilbert space is unitarily isomorphic to the Schrödinger representation on $\mathbf{H} = L^2(\mathbb{R}^n)$. The Schrödinger representation $\rho(N)$ satisfies the following integrated form of the canonical commutation relations (2.1), known as the Weyl relations.

Let s, $t \in \mathbb{R}$ and let f and g be elements of X generating the unitary groups $\rho(tf)$ and $\rho(sg)$, respectively; then [32, p. 139]

$$\rho(tf)\, \rho(sg) = e^{(2ist/h)\omega(f,g)} \rho(sg)\, \rho(tf). \tag{2.2}$$

Let $Sp(T^*\mathbb{R}^n)$ be the group of linear symplectic automorphisms of $T^*\mathbb{R}^n$. Then Sp acts in a natural way as automorphisms of N and the Stone–von Neumann theorem gives a representation, which we will also denote by ρ, of Sp as C^*-algebra automorphisms of $\rho(N)$. By a theorem of Shale [30] this is the projection to Sp of a *unitary* representation

$$\rho\colon Mp \to U(\mathbf{H}),$$

where Mp is the metaplectic group, the double covering of Sp. One gets in this way a representation of the inhomogeneous metaplectic group, the semidirect product of N with Mp,

$$\rho\colon IMp = M_p \ltimes N \to U(\mathbf{H}). \tag{2.3}$$

This solves the quantization problem completely in the case of linear finite dimensional systems.

In the infinite dimensional case the situation is more subtle. We will not consider the most general situation possible but will confine ourselves to the case of a Hilbert space X with a strong symplectic form. Then X is isomorphic to a complex Hilbert space K (see [4]). The basic observables are here elements f of K considered as linear functionals and the classical dynamics is given by some one parameter group Q_t of linear symplectic transformations.

Here we can form the Heisenberg group in the same way as above and construct representations of it on some Hilbert space under quite general conditions. It must be noted, however, that here there are many (unitarily) inequivalent representations. These are determined by, for example, a generating functional (which plays a role similar to the generating functional of a Kaehler metric in complex geometry) and certain additional information as, for example, uniqueness of the vacuum state, see [1]. In particular, there is no generalization of the Stone–von Neumann theorem; instead, there is the analogous result that the algebra of field observables is uniquely determined when considered as an abstract C^*-algebra [24, Theorem 1].

In order to get a fixed representation space one has to put stronger assumptions on the dynamics. According to a theorem by Cook [6] every sufficiently regular linear dynamical system determines an invariant complex structure and consequently, using the Fock–Cook representation determines a quantization. In this case it is possible to characterize completely the admissible symplectic transformations; see [30].

If we put even stronger conditions we get a result completely in parallel with the finite dimensional situation [30, Sect. 5.2]. Let N denote the Heisenberg group corresponding to K and let $Sp(K)_1$ denote the class of symplectic transformations of K of the form $I_K + T$ with T of trace class. The group $Sp(K)_1$ has a double covering which we denote by $Mp(K)_1$. There is a unitary representation

$$\rho\colon IM_p(K)_1 = Mp(K)_1 \ltimes N \to U(\mathbf{H})$$

from $IM_p(K)_1$, the semidirect product of N, the Heisenberg group corresponding to K, with $Mp(K)_1$ to the unitary operators on the representation space $\mathbf{H}$, which in this case is the Fock-space or equivalently can be described as $L^2(M, p)$, the space of square-integrable functions on a Lagrangian subspace M of K, with respect to the canonical Gaussian measure p corresponding to the induced inner product on M.

After this aside on the linear theory we will now state the definitions concerning prequantization which will be used in the rest of the paper. As remarked in Section 1, we will need to define prequantization w.r.t. a volume form μ which is not invariant under the relevant transformations. We will consider here only the formal aspects of the problem. The rest of the paper will be devoted to carrying out the construction sketched here.

Let (B, ω) be a symplectic Banach space and let μ be a (sufficiently regular) finite measure on B. Consider measurable mappings Φ, $\Psi\colon B \to B$.

We can define the pull-back Φ_μ^* of μ by Φ, by letting $\Phi^*\mu$ be the measure such that

$$(\Phi^*\mu)(E) = \mu(\Phi(E)),$$

for any measurable set E. If it is true that $\Phi^*\mu$ is equivalent to μ (we call such mappings μ-admissible), then it is possible to define a Jacobian function J_Φ: $B \to \mathbb{R}$ by the Radon–Nikodym derivative

$$J_\Phi(x) = \frac{d\Phi^*\mu}{d\mu}(x).$$

Formally, the Jacobian functions satisfy the algebraic relations

$$J_{\Phi \circ \Psi}(x) = J_\Phi(\Psi(x))\, J_\Psi(x),$$

which allows one to define a line bundle W_μ: $B \times \mathbb{R} \xrightarrow{\pi} B$ (cf. [8]). Since W_μ is a real line bundle it is possible to define bundles W_μ^r for $r \in \mathbb{R}$ using J_Φ^r. W_μ is trivial and there is a global section $1 = (x, 1)$ so we can write a general section as $s = \bar{s}(x)\, 1$, where $\bar{s}$ is some function. In particular, the closure of the space of smooth (in some appropriate sense) section of $W_\mu^{1/2}$ becomes a Hilbert space $\mathbf{H}$ with the inner product

$$\langle s_1, s_2 \rangle_{\mathbf{H}} = \int_B \bar{s}_1 \bar{s}_2\, d\mu. \tag{2.4}$$

On $\mathbf{H}$ there is a natural representation of the group of μ-admissible transformations as unitary operators, given by the lifting of Φ to $\mathbf{H}$,

$$\tilde{\Phi} s = J_\Phi^{1/2}(\Phi^* \bar{s})\, 1, \tag{2.5}$$

which is easily seen to preserve the inner product (2.4) since

$$\langle \tilde{\Phi} s_1, \tilde{\Phi} s_2 \rangle_{\mathbf{H}} = \int_B \Phi^*(\bar{s}_1 \bar{s}_2)\, J_\Phi\, d\mu = \int_B \bar{s}_1 \bar{s}_2\, d\mu = \langle s_1, s_2 \rangle_{\mathbf{H}}.$$

Now, let ω be some symplectic form on B with symplectic potential ϑ, $\omega = d\vartheta$ and let $\mathbf{F}$ be a suitable set of real functions on B such that the Hamiltonian vector field X for any $f \in \mathbf{F}$ w.r.t. ω is such that X generates a global flow on B and such that $X \rfloor \vartheta$ is a continuous function. Then for $f \in \mathbf{F}$, it is possible to lift the corresponding Hamiltonian flow Φ_t to the "prequantum line bundle" Λ, i.e., a Hermitian line bundle with connection ∇ such that $\operatorname{curv}(\nabla) = (1/h)\omega$. This lifting is given by (cf. [32, 5.6.10]),

$$\hat{\Phi}_t = (\Phi_t^* \bar{s})\, e^{(i/h) \int_0^t L_f(x_t)\, dt}, \tag{2.6}$$

where $x_t = \Phi_t(x)$ and L_f is given by $X_f \rfloor \vartheta - f$. This gives a representation of the group of μ-admissible symplectic automorphisms of B as bundle automorphisms of Λ which preserve the Hermitian structure. In order to get a unitary representation we have to introduce an inner product on the space of sections. We do this by considering, instead of Λ, the bundle $W_\mu^{1/2} \otimes \Lambda$ and taking the representation space $\mathbf{H}$ to be the completion of the space of sections of $W_\mu^{1/2} \otimes \Lambda$, w.r.t. the inner product (let $s_i = \bar{s}_i 1_W \otimes 1_\Lambda$, $i = 1, 2$),

$$\langle s_1, s_2 \rangle_{\mathbf{H}} = \int_B \langle \bar{s}_1, \bar{s}_2 \rangle_\Lambda \, d\mu.$$

wgere $\langle \ , \ \rangle_\Lambda$ denotes the Hermitian metric on Λ. The lifting corresponding to (2.5) and (2.6) becomes

$$\rho(\Phi_t) s = J_{\Phi_t}^{1/2}(\Phi^* \bar{s}) \, e^{(i/h) \int_0^t L_f(x_t) \, dt}. \tag{2.7}$$

We can now easily derive that the generator $\rho(X)$ of the unitary one parameter group $\rho(\Phi_t)$ should be given by (1.2′). To see that the representation ρ satisfies the canonical commutation relations (Q1) one simply notes that δ_μ satisfies $\delta([X, Y]) = X\,\delta(Y) - Y\,\delta(X)$ and a simple calculation gives (Q1). This completes the sketch of the heuristic idea that this paper makes rigorous in the case where μ is a Gaussian measure.

3. Abstract Wiener Spaces

The concept of an Abstract Wiener Space is a generalization, introduced by Gross [12], of the construction by Wiener of a Gaussian measure on the space $C([0, 1])$ of continuous functions on the interval. The category of Abstract Wiener Spaces seems to be a natural setting for analysis on infinite dimensional spaces. There are generalizations of, for example, harmonic analysis [13], exterior algebra [21] and distribution theory [17].

For the convenience of the reader, we here collect the basic notations and definitions concerning Abstract Wiener Spaces. See [12] or [23] for details.

Definition 3.1. An Abstract Wiener Space (AWS for short) consists of a triple (B, H, i), where B (with norm $\| \ \|_B$) is a separable Banach space, H (with norm $\| \ \|_H$) is a separable Hilbert space and i: $H \to B$ is a linear imbedding of H into B such that $i^* \| \ \|_B$ is a measurable norm w.r.t. the canonical Gaussian Measure on H.

We will here always have $H \subset B$ and $i =$ the inclusion map. The canonical Gaussian measure on H with variance parameter t extends uniquely to a σ-additive Borel-measure p_t on B. Since changing the variance parameter causes only trivial changes in the results we will assume throughout this paper that $t = 1$ and denote p_t by p.

The class of AWS we will use in this paper is provided by the following:

PROPOSITION 3.1. *Let M be a compact C^∞ manifold of dimension n and let $L_s^2(M)$ denote the s-Sobolev space of L^2-type over M. Then, if*

$$t - s > n/2$$

the triple (L_s^2, L_t^2, i), where i is the inclusion map given by the Sobolev Imbedding Theorem, is an AWS.

For a discussion of this, see [9]. With the above definitions (B, p) is a finite measure space and we can now introduce standard concepts such as a.e.-convergence, $L^2(B, p)$, $L^2_{\text{loc}}(B, p)$, etc.

A concept which is central in analysis on Abstract Wiener Spaces is H-differentiability, a notion of differentiability which is weaker than Frechet-differentiability and which is natural for analysis on AWS. Let (B, H, i) be an AWS (with $H \subset B$), let E be some Banach space and let F: $O \to E$ be a mapping defined on an open subset O of B. For a fixed $x \in O$, let $G(h) = F(x + h)$ be defined for $h \in H$ such that $x + h \in O$. Then G is defined on a neighborhood of the origin in H.

DEFINITION. F is called H-differentiable at x if G is Frechet differentiable at the origin. Then $G'(0)$, the Frechet derivative of G at 0, is in $L(H, E)$, the space of continuous linear maps from H to E and we define the H-derivative $DF(x)$ of F at x by $DF(x) \equiv G'(0)$. F is said to be in $H - C^1(O, E)$ if

(H-D1) F is H-differentiable at each $x \in 0$;

(H-D2) F is continuous from O to E;

(H-D3) the map $x \to DF(x)$: $O \to L(H, E)$ is continuous.

F is said to be in $H - C^2(O, E)$ if F and DF are in $H - C^1(O, E)$ and $H - C^1(O, L(H, E))$, respectively. $H - C^k$ and $H - C^\infty$ are defined inductively and we will take $H - C^0$ to mean continuous mappings.

Remark. Note that H-differentiability is considerably weaker than Frechet differentiability. Indeed, without the continuity assumptions in the above definition such mappings may fail to be even Borel-measurable. See [21, pp. 282–283] for a discussion of this.

Let (B, H, i) be an AWS and let $HS(H)$ denote the space of

Hilbert–Schmidt operators on H. It will be useful to introduce another, slightly stronger notion of differentiability (this will be motivated by the concept of admissible mapping introduced in Section 4).

DEFINITION 3.3. Let O be an open subset of B and let $F\colon O \to B$ be such that $F(0) \subset H$ and $DF(x) \in HS(H)$, for all $x \in O$ and the mapping $O \to H \times HS(H)$ defined by $x \to (F, DF)$ is of class $H-C^k$. Then F is said to be in $A^k(O)$.

Even though we are working on a linear space and can identify sections of TB with maps $B \to B$ it will be useful to introduce, following [21], the H-tangent bundle of B, $TH(B)$ which in the linear case can be identified with $B \times H$. We have naturally defined the spaces $H-C^k(TH(O))$ of $H-C^k$ sections over O and $A^k(TH(O))$ of A^k-sections over O, an open subset of B.

Both $H-C^k$ and A^k have natural topologies defined by the locally uniform topology. This makes $H-C^k$ and A^k into Frechet spaces.

4. THE JACOBI THEOREM

We will now state the result which is the starting point for the investigations in this paper, the Jacobi theorem for nonlinear transformations of an AWS. The version which is used here, due to Ramer [23], is the strongest to date for the case of nonlinear transformations and is, although it does not appear to be the strongest possible, necessarily nearly so, since specializing to the linear case one easily recovers the well-known result due to Segal and Feldman which is known to be the strongest possible. The problem of giving *necessary* conditions for a nonlinear transformation Φ to satisfy $\Phi^* p \leqslant p$ is quite subtle and has not been completely solved. See the discussion in [10].

To state the Jacobi theorem we need the following notion:

DEFINITION 4.1. Let $T \in HS(H)$, let $\{\lambda_i\}_{i=1}^{\infty}$ be the eigenvalues of T and let I_H be the identity transformation of H to itself. Then we define the Carleman–Fredholm determinant [31] $\chi(I_H + T)$ of $I_H + T$ to be

$$\chi(I_H + T) = \prod_{i=1}^{\infty} (1 + \lambda_i)\, e^{-\lambda_i}.$$

Now let $T\colon O \to B$ be in A^0 (see Definition 3.3). Then, as shown in [23, Lemma 4.3] there is a well-defined random variable (i.e., measurable function) which is denoted by

$$\text{“}\langle T(x), x\rangle_H - \operatorname{tr}(DT(x))\text{.”} \tag{4.1}$$

Here DT is the H-derivative of T as defined in Section 3. Note that the assumption that $T \in A^0$ only means that $DT \in HS(H)$ and so $\mathrm{tr}(DT)$ is not defined by itself. We denote the space of trace-class operators on H by $N(H)$.

We are now ready to state the Jacobi theorem for nonlinear transformations of Gaussian measures.

THEOREM 4.1 (Ramer [23]). *Let (B, H, i) be an AWS and let p be the induced Gaussian measure on B with variance parameter* 1. *Let O be an open subset of B and $\Phi = I + T$: $O \to B$ a continuous map such that*

(1) *Φ is a homeomorphism of O onto an open subset of B.*

(2) *$T \in A^0(O)$ and $I_H + DT(x) \in GL(H)$ for each $x \in O$.*

Then p and $\Phi^* p$ are mutually absolutely continuous as measures on O. The Radon–Nikodym derivative of $\Phi^* p$ w.r.t. p is given by

$$\frac{d\Phi^* p}{dp}(x) = J_\Phi(x),$$

where

$$J_\Phi(x) = |\chi(D\Phi(x))| \, e^{-\text{``}\langle T(x), x\rangle_H - \mathrm{tr}(DT(x))\text{''} - 1/2 \, \|T(x)\|_H^2}. \tag{4.2}$$

When $DT(x) \in N(H)$ for all $x \in B$ and $\langle T(x), x\rangle_H$ exists in the usual sense, then $\det(D\Phi(x))$, *the Fredholm determinant of $D\Phi(x)$ also exists and we have*

$$J_\Phi(x) = |\det(D\Phi(x))| \, e^{-T\langle T(x), x\rangle_H - 1/2 \, \|T(x)\|_H^2}. \tag{4.3}$$

DEFINITION 4.2. A nonlinear transformation Φ satisfying the assumptions of the above theorem is said to be *admissible* and we denote this by $\Phi \in \mathrm{Diff}_2(B)$ (analogously to the notion used in [30]). If in addition the mapping T is in $A^k(O)$ for some $k \in 0, ..., \infty$ we say that Φ is *k-admissible.* The class of k-admissible transformations of B is denoted by $\mathrm{Diff}_2^k(B)$. The J_Φ defined above is called the *Jacobian* of Φ w.r.t. p.

The following algebraic properties of the Jacobian functions will be useful:

LEMMA 4.1. *The class of k-admissible transformations is closed under inverses and under compositions. Let Ψ: $O \to B$ be k-admissible and assume that Φ: $\Psi(0) \to B$ is k-admissible. Then $\Phi \circ \Psi$ is k-admissible and*

$$J_{\Phi \circ \Psi}(x) = J_\Phi(\Psi(x)) \, J_\Psi(x); \qquad \text{a.e.} \quad \text{on } O. \tag{4.4}$$

Proof. The relation (4.4) is proved in [23, Lemma 5.4]. The closedness under inverses and compositions of the class of admissible mappings is proved in [21, Lemma 3]. Let $\Phi = I_B + T$ be admissible. In the proof of Lemma 3 in [21] it is shown that $T' = \Phi^{-1} - I_B$ satisfies

$$DT'(x) = -DT(\Phi^{-1}(x)) \circ [I_H + DT(\Phi^{-1}(x))]^{-1}.$$

We have here corrected the small error occurring in the calculation in [21]. From this expression, it is clear that Φ^{-1} is admissible and further, that if $\Phi \in \mathrm{Diff}_2^k(B)$ then the same holds for Φ^{-1}. Similarly, assume that $\Phi_i = I_B + T_i$ $(i = 1, 2)$ are k-admissible. Then

$$\Phi_1 \circ \Phi_2 = I_B + T_2 + T_1 \circ (I_B + T_2),$$

which is easily seen to be in $\mathrm{Diff}_2^k(B)$. ∎

Eells and Elworthy [8] have pointed out that for admissible Φ, J_Φ transform as the transition functions of a real line bundle, the bundle of Wiener densities which in the case we are interested in here is trivial. Obviously, for any $r \in \mathbb{R}$, we have a corresponding bundle of r-densities with transition functions J_Φ^r.

We are here interested in the bundle of Wiener $\frac{1}{2}$-densities given by $J_\Phi^{1/2}$. These should not be confused with the $\frac{1}{2}$-densities occurring in Geometric Quantization which are constructed to be normal to a polarization. We can state the basic facts about Wiener $\frac{1}{2}$-densities on an AWS as follows.

THEOREM 4.2. *Let (B, H, i) be an AWS. Then there exists a trivial line bundle*

$$W^{1/2} = B \times \mathbb{R} \xrightarrow{\pi} B.$$

The bundle $W^{1/2}$ has a global section $1(x) = (x, 1)$. Let $s = \bar{s}(x)1$ be a section of $W^{1/2}$ and let Φ be admissible. Then, to Φ, there corresponds a transformation $\tilde{\Phi}$ of $W^{1/2}$ which acts on a section s by

$$\Phi s = J_\Phi^{1/2}(\Phi^* \bar{s})\, 1.$$

5. ONE-PARAMETER FAMILIES OF ADMISSIBLE TRANSFORMATIONS

Let $O \subset B$ be an open subset and let $\varepsilon > 0$ be given. Let $\{X_t \colon t \in [-\varepsilon, \varepsilon]\}$ be a one-parameter family of vector fields in $A^k(TH(O))$ which is continuous w.r.t. t on O and let Φ_t be the one-parameter family of maps generated by X_t. Let $\Phi_0 = I$.

LEMMA 5.1. *Let Φ_t, X_t be as above. Then there exists some $\tilde{\varepsilon} > 0$ such that Φ_t is k-admissible in O for each $t \in [-\tilde{\varepsilon}, \tilde{\varepsilon}]$.*

Proof. By the boundedness of $\|X_t\|_H$ which implies boundedness of $\|X_t\|_B$ it follows from standard ODE theory [18] that there exists an $\tilde{\varepsilon} > 0$ such that $\Phi_t\colon O \to B$ is a homeomorphism for $t \in [-\tilde{\varepsilon}, \tilde{\varepsilon}]$.

Thus we need only check that $D\Phi_t(x) \in GL(H)$, $x \in O$ and $T_t \in A^k(O)$ for all t in some interval. To prove the first statement, note that by the chain rule

$$\frac{\partial D\Phi_t}{\partial t}(x) = DX_t(\Phi_t(x))\, D\Phi_t(x).$$

By assumption, $\|DX_t\|_{HS(H)}$ is bounded so $\|DX_t\|_{L(H,H)}$ is also bounded and we can again apply standard ODE-theory to find that

$$\|I_H - D\Phi_t\|_{L(H,H)} < 1$$

and thus $D\Phi_t$ is invertible for small t. To prove the second statement note that $\Phi_t = I + T_t$ so

$$\frac{\partial \Phi_t}{\partial t} = \frac{\partial T_t}{\partial t}$$

and by the chain rule we get for $0 \leqslant j \leqslant k$

$$\frac{\partial}{\partial t} D^j T_t = D^j X_t(\Phi_t) \frac{\partial}{\partial t} D^j \Phi_t.$$

Thus $X_t \in A^k$ implies that $T_t \in A^k$. This completes the proof of Lemma 5.1. ∎

Let $X \in A^1(TH(O))$. Then by the above results we know that X generates a flow, which we will denote by Φ_X. From the proof of Lemma 5.1 we see that there is an $\varepsilon > 0$ such that $\Phi_{X,t}$ is admissible on O for every $t \in [-\varepsilon, \varepsilon]$. This means that $\Phi_{X,t}$ has a Jacobian $J_{\Phi_{X,t}}$ and writing

$$\Phi^*_{X,t}\, p = J_{\Phi_{X,t_t}}\, p$$

we get formally

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \Phi^*_{X,t}\, p = \delta(X)\, p$$

or

$$L_X\, p = \delta(X)\, p, \tag{5.1}$$

where L_X denotes the Lie derivative w.r.t. X. We shall prove in Section 6 that under the assumption $X \in A(TH(O))$ we have

$$\delta(X) = -\text{“}\langle X(x), x\rangle_H - \mathrm{tr}(DX(x))\text{”}, \tag{5.2}$$

where “$\langle X(x), x\rangle_H - \mathrm{tr}(DX(x))$” is the random variable defined by the formula (4.1). Further, since in general $L_{[X,Y]} = [L_X, L_Y]$, we have, again formally, the usual formula for the divergence of a bracket,

$$\delta([X, Y]) = X\,\delta(Y) - Y\,\delta(X), \tag{5.3}$$

which is crucial in defining the prequantization (1.2′).

6. The Divergence

Piech [21] has introduced a notion of exterior algebra for Abstract Wiener spaces and Wiener–Riemann manifolds (Banach manifolds with an AWS structure on each tangent space and an admissible atlas) which gives a natural setting for the geometry of these spaces. A fundamental step is the definition of an exterior derivative d and an exterior coderivative δ as the L^2-adjoint of the exterior derivative d. We shall here only be interested in these notions for H-differentiable functions and vector fields (sections of TH) on an AWS.

Let (B, H, i) be an AWS and let O be an open subset of B. Assume that $f \in H - C^1(O, R)$ and $X \in A^0(TH(O))$. Then

$$L_X f = \frac{\partial}{\partial t}\Phi^*_{X,t} f = \langle df, X\rangle_H.$$

Piech now defines δ as the adjoint of d w.r.t. the $L^2(B, p)$-inner product, i.e.,

$$\int_B \langle df, X\rangle_H\, dp = -\int_B f\,\delta(X)\, dp$$

(where we have inserted a minus sign for notational convenience) and shows that

$$\delta(X) = -\text{“}\langle X(x), x\rangle_H - \mathrm{tr}(DX(x))\text{”},$$

which under the above assumptions is locally an element of $L^2(B, p)$. The following simple continuity property of the map

$$T \to \text{“}\langle T(x), x\rangle_H - \mathrm{tr}(DT(x))\text{”}$$

will be useful.

LEMMA 6.1. *Let (B, H, i) be an AWS and let O be an open subset of B. Let $\{T_s\}_{s\in[-\varepsilon,\varepsilon]}$ be a one-parameter family of maps $O \to B$ such that $T_t \in A^0(O)$ for every $t \in [-\varepsilon, \varepsilon]$. Further assume that as the map $t \to T_t$ is continuous w.r.t. the $A^0(O)$-topology. Then the map*

$$t \to \text{“}\langle T_t(x), x\rangle_H - \operatorname{tr}(DT_t(x))\text{”} \in L^2_{\mathrm{loc}}(O)$$

is continuous.

Proof. By the linearity of the expression “$\langle T(x), x\rangle_H - \operatorname{tr}(DT(x))$” it is sufficient to consider the continuity at $t=0$ assuming that $T_0 \equiv 0$.

Choose $x_0 \in O$ and $r > 0$ such that $B(x_0, 2r)$, the ball in B with radius $2r$ centered at x_0, is contained in O. Next construct a function $h\colon B \to \mathbb{R}$ as in the proof of Lemma 4.3, on p. 177 of [23], with the following properties:

$$h(B) \subset [0, 1], \qquad h\,|_{B(x_0,r)} \equiv 1, \qquad h\,|_{B\setminus B(x_0,2r)} \equiv 0 \tag{6.1}$$

$$\exists \text{ a constant } c \text{ such that } \|dh(x)\|_{B^*} \leqslant c < \infty, \qquad \forall\, x \in B. \tag{6.2}$$

We now consider the random variable

$$\text{“}\langle (hT_t)(x), x\rangle_H - \operatorname{tr}[D(hT_t)(x)]\text{”},$$

which by construction coincides with “$\langle T(x), x\rangle_H - \operatorname{tr}(DT(x))$” on $B(x_0, r)$ and extend it to the whole of B. Applying an estimate due to Ramer [23, Lemma 4.1] gives the inequality

$$\int_B (\text{“}\langle (hT_t)(x), x\rangle_H - \operatorname{tr}[D(hT_t)(x)]\text{”})^2\, dp$$

$$\leqslant \int_B (\|hT_t(x)\|_H^2 + \|D(hT_t(x)\|_{HS(H)}^2)\, dp. \tag{6.3}$$

It follows easily from the properties (6.1) and (6.2) of h that the bound (6.3) tends to 0 as $t \to 0$, which implies that

$$\lim_{t\to 0} \|\text{“}\langle T_t(x), x\rangle_H - \operatorname{tr}(DT_t(x))\text{”}\|_{L^2(B(x_0,r))} = 0,$$

which since x_0 was arbitrary completes the proof. ∎

The following result characterizes $\delta(X)$ as the logarithmic derivative of p w.r.t. X.

LEMMA 6.2. *Let (B, H, i) be an AWS, let O be an open subset of B and let $\varepsilon > 0$. Assume that $\{\Phi_s\}_{s\in[-\varepsilon,\varepsilon]}$ is a one-parameter family of admissible* *transformations of O to B. Further asume that Φ_t is generated by a vector field X_t satisfying the assumptions in Lemma* 5.1. *Then the mapping*

$$t \to J_{\Phi_t}$$

is C^1 p-almost everywhere on O in the sense that there exists an a.e. defined measurable function $\delta(X_t)$ on O such that

$$\frac{\partial}{\partial t} J_{\Phi_t} = (\delta(X_t) \circ \Phi_t)\, J_{\Phi_t}, \qquad \text{a.e.} - p \tag{6.4}$$

on O. Here $\delta(X) = -\text{"}\langle X(x), x\rangle_H - \operatorname{tr}(DX(x))\text{."}$

Proof. First note that by Lemma 4.1, $1 = J_{\Phi \quad \Phi^{-1}} = J_\Phi(\Phi^{-1})\, J_{\Phi^{-1}}$ so we have

$$J_{\Phi_t} \circ \Phi_{t'}^{-1} - 1 = (J_{\Phi_t} - J_{\Phi_{t'}}) \circ \Phi_{t'}^{-1} J_{\Phi_t^{-1}}.$$

Thus we can, without loss of generality, assume that $t' = 0$ and $\Phi_0 = I$. Study, then

$$\text{a.e.} - \lim_{t \to 0} \frac{1}{t} (J_{\Phi_t} - 1).$$

Recall that J_{Φ_t} is given by

$$J_{\Phi_t} = \chi(D\Phi_t)\, e^{-\text{"}\langle T(x), x\rangle_H - \operatorname{tr}(DT_t(x))\text{"} - 1/2\, \|T_t\|_H^2}, \tag{6.5}$$

where $T_t = \Phi_t - I_B$. We will consider the derivative of each term of (6.5) separately and then use the chain rule. It is easily seen that

$$D_A \,|_{A=0}\, \chi(I + A) = 0,$$

which means that

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \chi(D\Phi_t) \equiv 0.$$

For the second term of (6.5) note that

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \tfrac{1}{2}\|T\|_H^2 = \langle X_0, T_0\rangle = 0$$

since by assumption $T_0 \equiv 0$. Further, by definition

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \text{"}\langle T_t(x), x\rangle_H - \operatorname{tr}(DT_t(x))\text{"}$$

$$= \lim_{t \to 0} \frac{1}{t} (\text{"}\langle T_t(x), x\rangle_H - \operatorname{tr}(DT_t(x))\text{"}). \tag{6.6}$$

Now, by assumption, $(1/t)T_t \to X_0$ in $A^0(O)$ as $t \to 0$. Thus by Lemma 6.1, the limit (6.6) exists a.e.-p and equals

$$\text{“}\langle X_0(x), x\rangle_H - \operatorname{tr}(DX_0(x))\text{”}.$$

Now we see that

$$\delta(X_0) = \frac{\partial}{\partial t}\bigg|_{t=0} - \text{“}\langle T_t(x), x\rangle_H - \operatorname{tr}(DT_t(x))\text{”}$$

and an easy manipulation gives (6.4). This completes the proof of Lemma 6.2. ∎

From this result we easily derive the expression (5.1). Using the definition of Lie derivative and the relation (6.4) with $t=0$ one gets

COROLLARY 6.1. *Let Φ_t and X_t be as in Lemma 6.2. Then*

$$L_{X_0} p = \delta(X_0)\, p.$$

These results now enable us to justify the formula (5.2).

THEOREM 6.1. *Assume that $X, Y \in A^1(TH(B))$. Then $[X, Y] \in A^0(TH(B))$ and $\delta([X, Y]) \in L^2_{\text{loc}}$. Further, we have the relation*

$$\delta([X, Y]) = X\,\delta(Y) - Y\,\delta(X), \qquad a.e.\text{-}p. \tag{6.7}$$

Proof. First note that by definition, we have

$$[X, Y] = DY \cdot X - DX \cdot Y \in A^0(TH(B)),$$

so by Lemma 6.1 with Φ_t as the flow of $[X, Y]$ we see that $\delta([X, Y])$ is well defined as an element of L^2_{loc}. Further, using the well-known formula $L_{[X,Y]} = [L_X, L_Y]$ and the relation $L_X p = \delta(X)\, p$ we get

$$\delta([X, Y])\, p = L_X(\delta(Y)\, p) - L_Y(\delta(X)\, p).$$

Thus, to prove the relation (6.7) we have to be able to differentiate $\delta(X)\, p$.

Now, $\delta(Y) \in L^2_{\text{loc}}$ implies $\delta(Y) \in L^1_{\text{loc}}$ so $\delta(Y)\, p$ is a local measure. To show that it is differentiable we write, locally

$$\begin{aligned} L_X(\delta(Y)\, p) &= \frac{\partial}{\partial t}\bigg|_{t=0} \Phi^*_{X,t}(\delta(Y)\, p) \\ &= \frac{\partial}{\partial t}\bigg|_{t=0} \Phi^*_{X,t}(\delta(Y))\, J_{\Phi_{X,t}}\, p, \end{aligned} \tag{6.8}$$

were we have used $\Phi^*_{X,t}\, p = J_{\Phi_{X,t}}\, p$. Since the t-derivative of each term in (6.8) exists a.e. and is in L^2_{loc} we see that

$$L_X(\delta(Y)\, p) = \delta(Y)\, \delta(X)\, p + (L_X \delta(Y))\, p$$

defines a local measure. Obviously

$$\delta(X)\, \delta(Y)\, p = \delta(Y)\, \delta(X)\, p.$$

so we get, writing $L_X\, \delta(Y) = X\, \delta(Y)$ and similarly for $L_Y\, \delta(X)$,

$$\begin{aligned} \delta([X, Y]) &= X\, \delta(Y) + \delta(Y)\, \delta(X) - Y\, \delta(X) - \delta(X)\, \delta(Y) \\ &= X\, \delta(Y) - Y\, \delta(X). \quad \blacksquare \end{aligned}$$

7. Almost Admissible Transformations

In our applications, we will need the following more general class of transformations:

Definition 7. Let (B, H, i) be an AWS and let Φ: $B \to B$ be a map such that

$$\Phi = Q \circ \Psi,$$

where Q is a continuous linear endomorphism of B such that

(i) Q leaves H invariant and $Q\,|_H$ is an isometry;

(ii) Ψ: $B \to B$ is an admissible transformation.

Then we say that Φ is *almost admissible* and denote this by $\Phi \in r\,\mathrm{Diff}(B)$. If in addition, $\Psi = I + T$ with T locally in A^k then we say that Φ is *almost k-admissible* and denote this by $\Phi \in r\,\mathrm{Diff}^k(B)$.

Note that the space of almost admissible transformations is a nonlinear analogue of the Restricted General Linear Group introduced by Shale [30]. Indeed, assume that $\Phi = Q \circ \Psi$ is almost admissible and linear. Then $\Phi = Q \circ (I + T)$, where $T\,|_H \in HS(H)$, so $\Phi\,|_H \in rGL(H)$ in the notation of [30, Definition 2.1]. Obviously this notion can be extended as in the linear case by replacing the class of Hilbert–Schmidt operators by, say, any symmetrically normed ideal of compact operators. However, since we are studying transformations of a Gaussian measure, the present class seems to be the proper object to study.

Since Q preserves $\| \ \|_H$ it follows, as is well known, that Q preserves p, i.e.,

$$\frac{dQ^*p}{dp} \equiv 1.$$

The class of almost admissible transformations have some simple properties which we record here for future use.

LEMMA 7.1. *Let $\Phi = Q \circ \Psi \in r\,\mathrm{Diff}(B)$. Then J_Φ, the Jacobian of Φ w.r.t. p satisfies*

$$J_\Phi(x) = J_\Psi(x).$$

Proof.

$$J_\Phi\, p = Q^* \Psi^* p = \Psi^* p = J_\Psi\, p. \quad \blacksquare$$

LEMMA 7.2. *The class of almost k-admissible transformations is closed under compositions and inverses.*

Proof. Assume that $\Phi = Q \circ \Psi \in r\,\mathrm{Diff}^k(B)$. By definition, $\Psi \in \mathrm{Diff}_2^k(B)$ and by Lemma 4.1 $\Psi^{-1} = I_B + T'$ is also k-admissible. Now note that

$$\Phi^{-1} = \Psi^{-1} \circ Q^{-1} = Q^{-1} \circ \Psi',$$

where $\Psi' = I_B + Q \circ T' \circ Q^{-1}$, which clearly is k-admissible using the fact that $HS(H)$ is a norm ideal in $L(H, H)$.

Let

$$\Phi_i = Q_i \circ \Psi_i \in r\,\mathrm{Diff}^k(B) \qquad (i = 1, 2).$$

Then if $\Psi_i = I_B + T_i$ we get that

$$\Phi_1 \circ \Phi_2 = Q_1 Q_2 \circ (I_B + T_2) + Q_1 \circ T_1 \circ Q_2 \circ (I_B + T_2),$$

from which it easily follows, with an argument similar to the above, that $\Phi_1 \circ \Phi_2 \in r\,\mathrm{Diff}^k(B)$.

An easy calculation similar to the above also gives

LEMMA 7.3. *Let Ψ be k-admissible and Φ almost k-admissible. Then $\Phi^{-1} \circ \Psi \circ \Phi$ is k-admissible.*

8. The Unitary Group Generated by an Almost Admissible Flow

Let Φ_t be a global almost k-admissible flow generated by X. We will study the lifting of the flow Φ_t to $W^{1/2}$, i.e., a bundle automorphism $\tilde{\Phi}_t$ of $W^{1/2}$ as defined in (2.5).

Recall that if $s = \bar{s}(x)\, 1$ is a section of $W^{1/2}$, then

$$\tilde{\Phi}_t s = J_{\Phi_t}^{1/2}(\Phi_t^* \bar{s})\, 1.$$

THEOREM 8.1. *Let* $\mathbf{H}$ *be the completion of* $A^\infty(W^{1/2})$ *w.r.t. the norm* $\|\ \|_{\mathbf{H}}$ *defined by*

$$\|s\|_{\mathbf{H}}^2 = \int_B |s|^2\, dp$$

and let $\Phi_t = Q_t \Psi_t$ *be a flow of almost admissible transformations. Assume that the* $Q_t|_H$ *form a strongly continuous one-parameter group of isometries and that* $T_t = \Psi_t - I$ *satisfies the assumptions of Lemma* 6.1 *on* B. *Then* $\tilde{\Phi}_t$: $\mathbf{H} \to \mathbf{H}$ *is a strongly continuous group of unitary operators.*

Proof. We will think of the sections of $W^{1/2}$ as functions in $L^2(B, p)$ and write

$$(\tilde{\Phi}_t f)(x) = J_{\Phi_t}^{1/2}(x)\, f(\Phi_t(x)).$$

Let $\langle\ ,\ \rangle_{\mathbf{H}}$ be the inner product corresponding to the norm $\|\ \|_{\mathbf{H}}$ and let $f, g \in \mathbf{H}$. First note that $\tilde{\Phi}_t$ is unitary for each t since

$$\langle \tilde{\Phi}_t f, \tilde{\Phi}_t g\rangle_{\mathbf{H}} = \int_B (\tilde{\Phi}_t f)(\tilde{\Phi}_t g)\, dp = \int_B \Phi_t^*(fg)\, J_{\Phi_t}\, dp,$$

which, by the Jacobi theorem (theorem 4.1) equals

$$\int_B fg\, dp = \langle f, g\rangle_{\mathbf{H}}.$$

The group properties $\tilde{\Phi}_t^{-1} = \tilde{\Phi}_{-t}$ and $\tilde{\Phi}_{s+t} = \tilde{\Phi}_s \circ \tilde{\Phi}_t$ follow from the transformation rules of the Jacobian (Lemma 4.1).

It remains to check that the group $\tilde{\Phi}_t$ is strongly continuous. By the group property it is sufficient to consider

$$\lim_{t \to 0} \|(\tilde{\Phi}_t - I) f\|_{\mathbf{H}} = 0, \qquad \forall f \in \mathbf{H}.$$

From [30, Lemma 3.1] it follows easily that if Q_t is a strongly continuous group of isometries of H then Q_t lifts to a strongly continuous group $\tilde{Q}_t$ of unitary transformations of $\mathbf{H}$. Now using the fact that we can write $\tilde{\Phi}_t = \tilde{\Psi}_t \tilde{Q}_t$ (note order!) we get, with $\tilde{f}_t = \tilde{Q}_t f$,

$$\begin{aligned}
\|(\tilde{\Phi}_t - I) f\|_{\mathbf{H}} &= \|\tilde{\Psi}_t \tilde{f}_t - f\|_{\mathbf{H}} \\
&\leqslant \|(\tilde{\Psi}_t - I) f\|_{\mathbf{H}} + \|\tilde{\Psi}_t(\tilde{f}_t - f)\|_{\mathbf{H}} \\
&\leqslant \|(\tilde{\Psi}_t - I) f\|_{\mathbf{H}} + \|(\tilde{Q}_t - I) f\|_{\mathbf{H}},
\end{aligned}$$

so it is sufficient to prove that

$$\lim_{t \to 0} \|(\tilde{\Psi}_t - I) f\|_{\mathbf{H}} = 0, \qquad \forall f \in \mathbf{H}.$$

Since each $\tilde{\Psi}_t$ is unitary, we only have to consider $f \in D$, some dense subset of $\mathbf{H}$, which we here take to be the subset of bounded continuous functions in $L^2(B, p)$.

Using Lemma 6.1 we see that as t tends to 0,

$$\text{“}\langle T_t(x), x\rangle_H - \operatorname{tr}(DT_t(x))\text{”} \to 0, \qquad \text{a.e.-}p$$

and it is clear that $\|T_t\|_H$ tends to 0 locally uniformly. This gives immediately that

$$J_{\Psi_t} \to 1, \qquad \text{a.e.-}p$$

and since also $\Psi_t(x) \to x$ as $t \to 0$ for all $x \in B$ we can apply the Egorov theorem to get an increasing sequence $\{V_j\}_{j=1}^{\infty}$ of compact measurable subsets of B such that

(i) $\lim_{j \to \infty} p(V_j) = 1$;

(ii) for each j, $J_{\Psi_t} \to 1$ and $\|\Psi_t(x) - x\|_B \to 0$ uniformly on V_j.

Now let f be a continuous element of $L_2(B, p)$ and write

$$\|\tilde{\Psi}_t f - f\|_{\mathbf{H}}^2 = \int_{V_j} (\tilde{\Psi}_t f - f)^2 \, dp + \int_{B \setminus V_j} (\tilde{\Psi}_t f - f)^2 \, dp. \tag{8.1}$$

We will consider each term of (8.1) separately. First note that

$$\begin{aligned}
\tilde{\Psi}_t f - f &= J_{\Psi_t}^{1/2}(f \circ \Psi_t) - f \\
&= (J_{\Psi_t}^{1/2} - 1) f + J_{\Psi_t}^{1/2}(f \circ \Psi_t - f).
\end{aligned}$$

Using the uniform convergence of $J_{\Psi_t}^{1/2}$ and Ψ_t and the continuity of f it follows easily that

$$\lim_{t\to 0} \|\tilde{\Psi}_t f - f\|_{L^2(V_j, p)} = 0. \tag{8.2}$$

We now have to check the convergence on the complement of V_j. Let $D = \sup_{x\in B} |f(x)| < \infty$. Then

$$\begin{aligned}\|\tilde{\Psi}_t f\|^2_{L^2(B\setminus V_j, p)} &= \int_{B\setminus V_j} J_{\Psi_t} |f(\Psi_t)|^2 \, dp \\ &\leqslant D^2 \int_{B\setminus V_j} J_{\Psi_t} \, dp.\end{aligned}$$

By the uniform convergence of $J_{\Psi_t} \to 1$ on V_j, we have that

$$\|\tilde{\Psi}_t f\|^2_{L^2(B\setminus V_j, p)} \leqslant D^2 p(B\setminus V_j) + \varepsilon(t)$$

for some $\varepsilon(t) \to 0$ as $t \to 0$. Expanding $\|\tilde{\Psi}_t f - f\|^2_{L^2(B\setminus V_j, p)}$ using the triangle and Schwarz inequalities one finds

$$\|\tilde{\Psi}_t f - f\|^2_{L^2(B\setminus V_j, p)} \leqslant 4D^2 p(B\setminus V_j) + \varepsilon(t)$$

for some $\varepsilon(t) \to 0$ as $t \to 0$. Combining this with (8.2) and the fact that $p(B\setminus V_j) \to 0$ as $j \to \infty$ completes the proof. ∎

9. The Nonlinear Klein–Gordon Equation

As an application of the preceding theory in a concrete case we consider the simplest scalar relativistic field equation on a (Lorenz) space-time N

$$\Box u - m^2 u = -F'(u). \tag{9.1}$$

In order to be able to apply Proposition 3.1 to get an AWS-setting for (9.1) it is necessary to work on a spatially compact spacetime N, which in order to avoid the subtle problems concerning quantization of fields over non-stationary space times we define as $N = \Omega \times \mathbb{R}$, where Ω is some compact, n-dimensional Riemannian C^∞ manifold.

It should be noted that the assumption that Ω is compact amounts to a "space-cutoff" (with periodic boundary conditions) and that one might consider other boundary conditions. Following [4] we write Eq. (9.1) as a Hamiltonian system. First we introduce some notation. Let

$$C=(-\Delta+m^2),$$

where Δ is the Laplace–Beltrami operator on Ω.

DEFINITION 9.1. Let $s>0$. On $L_s^2(\Omega)$, define a norm $\|\ \|_s$ by

$$\|f\|_s^2=\langle C^s f, f\rangle_0, \tag{9.2}$$

where $\langle\ ,\ \rangle_0$ denotes the L^2-inner product on Ω w.r.t. g. In the following L_s^2 will always denote $L_s^2(M)$ with the norm given by (9.2). Let $r, s\in\mathbb{R}$ and let $L_r^2\times L_s^2$ be given the product norm w.r.t. the norm defined by (9.2). We denote this norm by $\|\ \|_{r,s}$ and the corresponding inner product by $\langle\ ,\ \rangle_{r,s}$.

Define a linear operator A on $L_{s+1}^2\times L_s^2$ by

$$A=\begin{pmatrix}0 & I\\ -C & 0\end{pmatrix}.$$

It can be shown that A is skew-adjoint on $L_{s+1}^2\times L_s^2$ and that consequently there is a one-parameter group

$$Q_t=e^{At}$$

of isometries of $L_{s+1}^2\times L_s^2$. On $L_1^2\times L_0^2$, define a (weak) symplectic form ω by

$$\omega(x, y)=\langle A^{-1}x,\ y\rangle_{1,0}. \tag{9.3}$$

Write the equation (9.1) as a system of ODEs:

$$\frac{\partial}{\partial t}\Phi_t=(A+Y)\circ\Phi_t,$$

where Y is the vector field given by

$$Y(x)=\begin{pmatrix}0\\ -F'(x_1)\end{pmatrix}, \tag{9.4}$$

where x_1 denotes the first component of x. Then $A+Y$ is a Hamiltonian vector field w.r.t. ω for the Hamiltonian

$$K(x)=\tfrac{1}{2}\|x\|_{1,0}^2+\int_\Omega F(x_1)\,dz. \tag{9.5}$$

Here, in order to avoid problems with the regularity of the solutions of the Klein–Gordon equation, we assume that $F\colon \mathbb{R} \to \mathbb{R}$ is C^∞ and non-negative and that $F(0) = F'(0) = 0$. We will also, for $n > 1$, need to consider a regularized version of the Klein–Gordon equation, defined by the Hamiltonian K_R defined by

$$K_R(x) = \tfrac{1}{2} \| x \|_{1,0}^2 + \int_\Omega F(Rx_1)\, dz, \tag{9.6}$$

where $R\colon L_1^2 \to C^\infty$ is some smoothing operator. This corresponds to the momentum cutoff usually introduced before renormalization is carried out. We denote the corresponding Hamiltonian vector field by $X_R = A + Y_R$ where Y_R is given by

$$Y_R(x) = \begin{pmatrix} 0 \\ -F(Rx_1) \end{pmatrix}.$$

We now introduce the AWS-setting for the Klein–Gordon equation. Let $s > n/2$ and define (B, H, i) by

$$\begin{cases} B = L_1^2 \times L_0^2 \\ H = L_{s+1}^2 \times L_s^2 \\ i = inclusion. \end{cases} \tag{9.7}$$

Then (B, H, i) is an AWS by Proposition 3.1. Let

$$\Psi_t = Q_t^{-1} \circ \Phi_t. \tag{9.8}$$

Note that Q_t is an isometry when restricted to H so we can apply Theorem 8.1 to get a unitary representation of Φ_t if $T_t = \Psi_t - I$ satisfies the assumptions of Lemma 6.1. The formula (9.8) is known as the energy representation, and it is well known that Ψ_t satisfies the differential equation

$$\frac{\partial}{\partial t} \Psi_t = Y_t \circ \Psi_t. \tag{9.9}$$

where $Y_t = Q_t^{-1} \circ Y \circ Q_t$ (cf. [20]). In order for the Klein–Gordon flow to be almost admissible it is necessary for Ψ_t to be admissible, which requires at least that $DY_t(x) \in HS(H)$ for all $x \in B$. The following lemma covers the case where the dimension n of Ω equals 1 (one space dimension), which is the only case which we will be able to handle without any momentum cutoffs.

LEMMA 9.1. *Let $n=1$ and let $s \in \mathbb{R}$ be such that $1/2 < s < 1$. Let $\varphi \in L_1^2$ and let*

$$\varphi(M): L_{s+1}^2 \to L_s^2$$

denote the operator of multiplication by φ. Then the mapping Φ of $H = L_{s+1}^2 \times L_s^2$ to itself defined by

$$\Phi: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 \\ \varphi(M)x \end{pmatrix}, \tag{9.10}$$

where (x, y) is some element of $L_{s+1}^2 \times L_s^2$, is in $HS(H)$. Further, the mapping $\varphi \to \Phi$ of L_1^2 to $HS(H)$ is continuous.

Proof. Writing Φ as a 2×2 matrix of operators we have

$$\Phi = \begin{pmatrix} 0 & 0 \\ \varphi(M) & 0 \end{pmatrix}.$$

In order to check whether Φ is an element of $HS(H)$ (i.e., $\mathrm{tr}(\Phi^*\Phi)$, the trace of $\Phi^*\Phi$ is finite), we need to compute the dual Φ^* of Φ. In matrix form Φ^* becomes

$$\Phi^* = \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}, \tag{9.11}$$

where Z is some operator from L_s^2 to L_{s+1}^2 which satisfies

$$\langle \varphi(M)x, y \rangle_s = \langle x, Zy \rangle_{s+1}$$

for any $x \in L_{s+1}^2$ and $y \in L_s^2$. Rewriting the left-hand side of (9.11) using Definition 9.1 and the self adjointness of C one easily gets

$$Z = C^{-(s+1)} \varphi(M) C^s. \tag{9.12}$$

We now wish to estimate $\mathrm{tr}(\Phi^*\Phi)$. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for L_{s+1}^2 consisting of eigenvectors of C,

$$Ce_i = \lambda_i e_i.$$

By definition and using the fact that $\Phi^*\Phi$ annihilates the second component of $L_{s+1}^2 \times L_s^2$, we have

$$\mathrm{tr}(\Phi^*\Phi) = \sum_{i=1}^{i=\infty} \langle \Phi^*\Phi e_i, e_i \rangle_{s+1}. \tag{9.13}$$

Writing a term in the sum explicitly we have, using Definition 9.1,

$$\begin{aligned}\langle \Phi^*\Phi e_i, e_i\rangle_{s+1} &= \langle \varphi(M)\, C^s\varphi(M)e_i, e_i\rangle_0\\ &= \langle C^s\varphi(M)e_i, \varphi(M)e_i\rangle_0\\ &= \|\varphi(M)e_i\|_s^2 \leqslant \|\varphi\|_1^2\, \|e_i\|_1^2,\end{aligned}$$

where we have used (9.12) and the fact that $e_i \in C^\infty(\Omega)$ and that L_r^1 is a Banach algebra for $r > n/2$ [7, Lemma 3.3.2]. Using the assumption that $\|e_i\|_{s+1} \equiv 1$ it is easy to see that $\|e_i\|_1^2 = \lambda_i^{-s}$, which gives

$$\|\Phi\|_{HS(H)}^2 \leqslant \operatorname{tr}(C^{-s})\, \|\varphi\|_1^2.$$

It is well known that for $n = 1$, tr (C^{-s}) is finite for any $s > \frac{1}{2}$ so the right-hand side of (9.13) is finite and $\Phi \in HS(H)$. The above estimates show that the linear mapping $\varphi \to \Phi$ is bounded and therefore also continuous. This completes the proof of Lemma 9.1. ∎

Using Lemma 9.1 we are now able to show that the Klein–Gordon flow is almost admissible.

THEOREM 9.1. *Let $n = 1$ and assume that $\frac{1}{2} < s < 1$. Let (B, H, i) be the corresponding AWS given by (9.7). Then the Klein–Gordon flow Φ_t consists of almost admissible transformations of B. Similarly, for $n > 1$ and $s > n/2$ the Hamiltonian flow corresponding to the regularized Hamiltonian K_R consists of almost admissible transformations of B.*

Proof. We wish to apply Theorem 4.1 to Ψ_t given by (9.8). It is well known that under the present assumptions, Φ_t is a global flow on B [26]. Further, for each t, Φ_t is a homeomorphism of B onto itself and in particular, the Frechet derivative $\Phi_t'(x) \in GL(B, B)$ for all $x \in B$ (due to the hyperbolicity of the Klein–Gordon equation). Further, by (9.8), $\Phi_t = Q_t \circ \Psi_t$ so $\Psi_t = Q_t^{-1} \circ \Phi_t$ is again a homeomorphism of B onto itself and $\Psi_t'(x) \in GL(B, B)$ for all $x \in B$. This implies immediately that the H-derivative $D\Psi_t(x) = \Psi_t'(x)\,|_H$ is in $GL(H, H)$ for all $x \in B$.

Let $T_t = \Psi_t - I_H$. Then T_t is a continuous mapping of $B \to H$. To see this, note that

$$\begin{aligned}\|T_t(x) - T_t(y)\|_H &\leqslant \int_0^t \|Y_\tau(\Psi_\tau(x)) - Y_\tau(\Psi_\tau(y))\|_H\, d\tau\\ &\leqslant \int_0^t \|Y(\Phi_\tau(x)) - Y(\Phi_\tau(y))\|_H\, d\tau,\end{aligned}$$

where we have used (9.9) and the fact that $Q_t|_H$ is an isometry. It is easy to see that $Y\colon B \to H$ is locally Lipschitz and combining this with the fact that

Φ_t is locally Lipschitz [26, Corollary 1.5] proves the continuity of $x \to T_t(x)$.

To show that $DT_t(x) \in HS(H)$ for all $x \in B$ and that $DT_t(x)$: $B \to HS(H)$ is continuous, we use the fact that $(\partial/\partial t)\, DT_t(x) = DY_t(\Psi_t(x)) \circ D\Psi_t$, which gives, using

$$DY_t(\Psi_t(x)) \circ D\Psi_t(x) = Q_t^{-1} \circ DY(\Phi_t(x)) \circ D\Phi_t(x).$$

the estimate

$$\| DT_t(x) - DT_t(y) \|_{HS(H)}$$
$$\leqslant \int_0^t \| DY(\Phi_\tau(x)) \circ D\Phi_\tau(x) - DY(\Phi_\tau(y)) \circ D\Phi_\tau(y) \|_{HS(H)}\, d\tau. \qquad (9.14)$$

From this follows directly that $DT_t(x) \in HS(H)$ for all $x \in B$. It is easily seen that $DY(x)$ is of the form (9.10) with $\varphi = F''(x_1)$. Because of the fact that L_1^2 is a Banach-algebra, the map $x \to F''(x_1)$ is continuous from B to L_1^2 and thus Lemma 9.1 together with the Lipschitz continuity of Φ_t and (9.14) implies the continuity of the map DT_t: $B \to HS(H)$. This completes the proof of Theorem 9.1 for the case $n = 1$. The proof for the case $n > 1$ and the regularized Hamiltonian K_R is similar. The regularization makes DT_t into a smoothing operator. ∎

We will now consider the Jacobian J_{Φ_t} of the Klein–Gordon flow and show that it has a simpler form than (4.1) and in particular that $\det(D\Phi_t)$ exists in a certain sense and is identically equal to 1. To this end, let $\{e_i\}_{i=1}^{\infty}$ be an ON-basis of L_{s+1}^2 and let $E_{2N} \subset L_{s+1}^2 \times L_s^2$ be the span of the vectors $\{f_j\}_{j=1}^{\infty}$ given by

$$f_i = (e_i, 0); \qquad 1 \leqslant i \leqslant N,$$
$$f_{N+i} = (0, e_i); \qquad N+1 \leqslant i \leqslant 2N,$$

Let the orthogonal projection onto E_{2N} be denoted by P_{2N}. Using the definition of trace, it is easy to prove

LEMMA 9.2. *Let Φ be as in Lemma 9.1. Then,*

$$\operatorname{tr}(P_{2N}\Phi) = 0.$$

Now let Φ_t be the Klein–Gordon flow and let Ψ_t be given by (9.8). Then, according to Lemma 8.1, $J_{\Phi_t} = J_{\Psi_t}$. Recalling the expression (4.2) for J, one sees that the logarithm of the Jacobian is

$$\ln(J_{\Psi_t})(x) = \ln(\chi(D\Psi_t(x)) - \text{“}\langle T_t(x), x\rangle_H - \operatorname{tr}(DT_t(x))\text{”}$$
$$- \tfrac{1}{2} \| T_t(x) \|_H^2. \qquad (9.15)$$

Let $\{P_{2N}\}_{N=1}^{\infty}$ be as in Lemma 9.2. To simplify notation, if $A\colon H \to H$ is any linear mapping, we will denote $P_{2N}AP_{2N}$ by $A^{(2N)}$. The Carleman–Fredholm determinant $\chi(D\Psi_t(x))$ can be defined by

$$\lim_{N \to \infty} \det(D\Psi_t^{(2N)}(x))\, e^{-\operatorname{tr}(DT_t^{(2N)}(x))},$$

where $\det(D\Psi_t^{(2N)}(x))$ denotes the determinant of $D\Psi_t^{(2N)}(x)$ considered as a linear automorphism of E_{2N}. Consequently, the first two terms in the right-hand side of (9.15) can be defined by

$$\lim_{N \to \infty} \ln(\det(D\Psi_t^{(2N)}(x)) - \operatorname{tr}(DT_t^{(2N)}(x))$$
$$-\langle P_{2N}T_t(x), x\rangle_H + \operatorname{tr}(P_{2N}\, DT_t(x)), \tag{9.16}$$

where the limit is taken locally in $L^2(B, p)$ as in the proof of [23, Lemma 4.3]. For fixed N, the trace-terms in (9.16) cancel, since

$$\operatorname{tr}(P_{2N}\, DT_t(x)\, P_{2N}) = \operatorname{tr}(P_{2N}\, DT_t).$$

Thus, if it is possible to justify the limit of the term containing the determinant without using the regularization given by the Carleman–Fredholm determinant χ, then we get a Jacobian of the form (4.3). In the present case we are able to prove this directly, as follows.

First, note that by (9.9) and the chain rule,

$$\frac{\partial}{\partial t} D\Psi_t(x) = DY_t(\Psi(x)) \circ D\Psi_t(x),$$

where $DY_t(x) = Q_t^{-1} \circ DY(Qx) \circ Q_t$. Now, choose the basis $\{e_i\}_{i=1}^{\infty}$ used in Lemma 9.2 to consist of eigenfunctions of C. Then the subspaces E_{2N} are invariant under A and, consequently, the projections P_{2N} commute with $Q_t = e^{At}$. Using the invariance of the trace under similarities and Lemma 9.2 we immediately get that $\operatorname{tr}(DY_t^{(2N)}) \equiv 0$, independently of N. Combining this with the fact that the determinant satisfies

$$\frac{\partial}{\partial t} \det(D\Phi_t^{(2N)}) = \operatorname{tr}(DY_t^{(2N)}(\Psi)) \det(D\Psi_t^{(2N)}),$$

we get $\det(D\Psi_t^{(2N)}) \equiv 1$, independently of N. We have thus proved that the limit

$$\lim_{N \to \infty} \langle P_{2N}T_t(x), x\rangle_H,$$

exists a.e. in B and thus defines a random variable which we will denote by "$\langle T_t(x), x\rangle_H$".

Remark. It should be noted that a priori, there is no reason why $\langle T_t(x), x \rangle_H$ should be even measurable on B, and therefore, the above argument should be interpreted as showing that $\langle T_t(x), x \rangle_H$ has a "stochastic extension" to B in the same sense as (4.1). Further, the fact that the flow of Y leaves invariant a Lagrangean subspace of B, namely the second component of $L^2_{s+1} \times L^2_s$, is crucial for the method we have used. It is this that allows the use of Lemma 9.2 in this situation.

Combining the above argument with Theorem 4.1 we get

THEOREM 9.2. *Let $n=1$ and assume that $\frac{1}{2} < s < 1$. Let (B, H, i) be the corresponding AWS given by (9.7). Then the Klein–Gordon flow Φ_t has a Jacobian of the form*

$$J_{\Phi_t}(x) = e^{-\text{``}\langle T_t(x), x \rangle_H\text{''} - 1/2 \, \| T_t(x) \|_H^2}.$$

If $n > 1$, a similar statement holds for the flow w.r.t. K_R.

Next we derive an expression for the divergence $\delta(X)$ of the vector field X generating the Klein–Gordon flow. Since Φ_t is almost admissible but not admissible, the work in Section 6 does not apply, but formally one would expect that $\delta(X) = \delta(A) + \delta(Y)$. The linear vector field A generates the transformation Q_t, which preserves p so it is natural that $\delta(A) = 0$ and thus $\delta(X) = \delta(Y)$. In fact, it is possible to prove this rigorously here. The notion that $-\delta$ is the $L^2(B, p)$-dual of the exterior derivative d leads us to define, for an appropriate subset of functionals $f\colon B \to \mathbb{R}$ such that $df(x) \in D(X)$, where $D(X)$ is the domain of the vector field X,

$$-\int_B f\, \delta(X)\, dp = \int_B \langle df, X \rangle_H \, dp = \int_B \langle df, A \rangle_H \, dp + \int_B \langle df, Y \rangle_H \, dp.$$

Now, $\langle df, A \rangle_H (x) = (\partial/\partial t)\,|_{t=0} f(Q_t x)$, so

$$\int_B \langle df, A \rangle \, dp = \frac{\partial}{\partial t}\Big|_{t=0} \int_B f(Q_t x)\, dp = \frac{\partial}{\partial t}\Big|_{t=0} \int_B f Q^*_{-t}\, dp = 0,$$

since Q_t preserves p. Thus, $\delta(A) = 0$ for skew-adjoint vector fields and we arrive at the expression $\delta(X) = \delta(Y)$. It should be noted that here we interpret $\delta(X)$ to exist in a distributional sense. Using the above and a method similar to the proof of Theorem 9.2 we arrive at

LEMMA 9.3. *Let $X = A + Y$ be as above. Then $\delta(X)$ is given by*

$$\delta(X) = -\text{``}\langle Y(x), x \rangle_H\text{''}, \tag{9.17}$$

where "$\langle Y(x), x\rangle_H$" is to be interpreted as a random variable defined by the limit

$$\lim_{N\to\infty} \langle P_{2N} Y(x), x\rangle_H.$$

Remark. Using the special form of Y, it can in fact be proved directly that "$\langle Y(x), x\rangle_H$" is a well-defined random variable. The nilpotency of DY also motivates that no trace-term appears in (9.17).

In order to carry out the prequantization as sketched in Section 2 we now introduce the appropriate line bundle. Since we are working on a linear space there is no obstruction to constructing a Hermitian line-bundle $\Lambda \to B$ with connection ∇ such that $\mathrm{curv}(\nabla)(1/h)\omega$. We simply take the trivial bundle $\Lambda = B \times \mathbb{C}$ and define ∇ by

$$\nabla_X s = Xs - \frac{i}{h}(X \rfloor \vartheta)s, \tag{9.18}$$

where ϑ is some symplectic potential for ω. Of course, since we are working on an infinite dimensional space, the boundedness of $\vartheta \rfloor X$ must be checked in each case, but it will turn out that in our example the natural candidates for ϑ are sufficiently regular so that $\vartheta \rfloor X$ is defined, even though the Hamiltonian vector field X for the Klein–Gordon flow is unbounded.

The representation space $\mathbf{H}$ will be the L^2 sections of $\Lambda \otimes W^{1\,2}$ with $W^{1\,2}$ as in Section 4 and the finite transformations will be given by the formula (2.7). For concreteness, we take ϑ to be the canonical 1-form on $TL^2(\Omega)$. Let (x_1, x_2) be coordinates on $TL^2(\Omega)$ and let $Z = (Z_1, Z_2)$ be some vector field. Then we can write ϑ as (cf. [4, Sect. 2.1])

$$Z \rfloor \vartheta(x) = -\langle x_2, Z_1(x)\rangle_0.$$

It is easy to see that, with $X = A + Y$ as above we have

$$X \rfloor \vartheta(x) = -\langle x_2, X_1(x)\rangle_0 = -\|x_2\|_0^2,$$

which is a continuous function on B and since $K(x)$ is also continuous on B, $L_K = X \rfloor \vartheta - K$ is continuous and thus, expressions like (2.7) are well defined for an appropriate class of sections.

We collect the results about the Klein–Gordon flow in the following:

THEOREM 9.2 (Pre-quantization of the Klein–Gordon flow). *Let $n = 1$ and let $\frac{1}{2} < s < 1$. Let (B, H, i, ω), ϑ and $\mathbf{H}$ be as above. Then, if K is the Klein–Gordon Hamiltonian defined in* (9.5), *the following statements are true*:

(1) *Φ_t, the Hamiltonian flow corresponding to K lifts to a strongly continuous group $\rho(\Phi)_t$ of unitary transformations of* $\mathbf{H}$ *given by* (2.7).

(2) *To K there corresponds a Hermitian operator $\rho(K)$ on* $\mathbf{H}$ *given by*

$$\rho(K) = -ihX - X \rfloor \vartheta - ih\tfrac{1}{2}\text{“}\langle Y(x), x\rangle_H\text{”} + K.$$

For $n > 1$ and $s > n/2$, similar statements are true, with K replaced by K_R.

Proof. Write the expression (2.7) as $\rho(\Phi_t) = \tilde{\Phi}_t \lambda_t$, where λ_t denotes the phase factor $\exp\{(i/h)\int_0^t L_k(x_t)\,dt\}$, where $x_t = \Phi_t(x)$. Since we are working over a linear space, we can take $\mathbf{H} = L^2(B, p) \otimes \mathbb{C}$, the space of complex-valued square integrable functions on (B, p), and let $\tilde{\Phi}_t$ act on $\mathbf{H}$ in the obvious way. By Theorem 9.1, $\tilde{\Phi}_t$ defines a unitary transformation of $\mathbf{H}$ for each t and by the continuity of K, it is clear that $\rho(\Phi_t)$ also is unitary ($|\lambda_t| \equiv 1$). Formally, it is also clear that $\rho(\Phi_t)$ forms a group.

To prove the strong continuity of $\rho(\Phi_t)$, we proceed as follows. Let $T_t = Q_t^{-1} \circ \Phi_t - I$. Then, similarly to the proof of Theorem 9.1, we can estimate

$$\|T_t(x)\|_H \leqslant \int_0^t \|Y(\Phi_t(x))\|_H\,dt$$

and

$$\|DT_t(x)\|_{HS(H)} \leqslant \int_0^t \|DY(\Phi_t(x)) \circ D\Phi_t(x)\|_{HS(H)}\,dt.$$

It follows that $\|T_t\|_H$ and $\|DT_t\|_{HS(H)}$ tend to 0 locally uniformly as $t \to 0$ and thus, that T_t satisfies the assumptions of Lemma 6.1. Therefore, Theorem 8.1 applies to show that $\tilde{\Phi}_t$ is a strongly continuous one parameter group.

From the definition of L_K, we can estimate that

$$\left|\int_0^t L_K(x_t)\,dt\right| \leqslant \int_0^t (\|\Phi_t(x)\|_B + K(\Phi_t(x))\,dt,$$

which immediately implies that $|\lambda_t - 1| \to 0$ locally uniformly. Let $f \in \mathbf{H}$. Then,

$$\begin{aligned}\|\rho(\Phi_t)f - f\|_{\mathbf{H}} &\leqslant \|\tilde{\Phi}_t\lambda_t f - \lambda_t f\|_{\mathbf{H}} + \|\lambda_t f - f\|_{\mathbf{H}}\\ &\leqslant \|\tilde{\Phi}_t - f\|_{\mathbf{H}} + \|\lambda_t f - f\|_{\mathbf{H}}.\end{aligned} \tag{9.19}$$

By the above, we know that the first term of (9.19) tends to 0 as $t \to 0$. By applying the Egorov theorem as in the proof of Theorem 8.1, using the fact that $\lambda_t \to 1$ locally uniformly and hence almost everywhere on (B, p), it is easy to prove that the second term of (9.19) also tends to 0. This completes the proof of part (1) of Theorem 9.3. Stone's theorem now applies to show that there exists a Hermitian operator $\rho(K)$ which one computes to be of the form given in (2).

For the case where $n > 1$, it is easy to see that similar methods apply to give analogous results for the regularized Hamiltonian K_R. ▮

Acknowledgments

This research was made possible by a Post-Doctoral grant from NFR (the Swedish Natural Science Research Council). The author is grateful to Stig I. Andersson for his hospitality at the department of Theoretical Physics in Clausthal-Zellerfeldt, where the work was started, and to Jerry Marsden for making possible a visit to Berkeley, where the work was completed. Thanks to Elmar Schrohe for some helpful discussions.

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